Behaviormetrika
1990, No. 27, 25-46

A GENERALIZED INNER PRODUCT MODEL FOR THE ANALYSIS OF ASYMMETRY

Naohito Chino*

A least squares procedure called GIPSCAL (a Generalized Inner Product multidimensional SCALing) is proposed which extends Chino's ASYMSCAL into higher dimensions than three. GIPSCAL fits the inner product of two vectors and the area of the parallelogram spanned by these vectors, respectively, for the symmetric and skew-symmetric parts of observed similarity judgements. It is shown that GIPSCAL has a very desirable property that the geometrical interpretation of asymmetric parts in similarity judgements is reducible to that of the area of the parallelogram spanned by vectors in two dimensions. It is also shown that GIPSCAL permits a social psychological justification for the cause of asymmetry. Relation to distance model is discussed. Examples of application are given to demonstrate the feasibility of the model.

1. Introduction

During the past twenty years, various multidimensional scaling (MDS) techniques which deal directly or indirectly with the asymmetric relationships among objects have been developed by many researchers.

The motivation of developing these asymmetric MDS's is, of course, to overcome a short-fall inherent in the traditional MDS techniques. That is, the traditional MDS's are not so powerful in cases when the similarity or dissimilarity data matrices are asymmetric by nature because they are based on the assumption that the similarity or dissimilarity data can be related to the interpoint distances in some metric space (Torgerson, 1958 ; Kruskal, 1964 ; Guttman, 1968).

For this reason, some researchers have augmented the traditional MDS's by the assumption that the similarity or dissimilarity between objects is a function not only of interpoint distance in a metric space but also of the quantities related to the objects. For example, the squared distance is augmented by differential weights for dimensions in the weighted Euclidean model proposed first by Young (Young, 1975, 1987 ; Baker, Young, and Takane, 1977 ; Hayashi, 1977). In Krumhansl's distance-density model, the interpoint distance is augmented by some measure of spatial density in the regions surrounding the points (Krumhansl, 1978). Weeks and Bentler (1980, 1982) and Saito (1983, 1986) have proposed modified distance models in which the interpoint distance is augmented by a few constants related to the points. Okada and Imaizumi (1984, 1987) have proposed a non-metric version of a generalized modified distance model.

Some other researchers have taken different approaches which utilize metric spaces. For example, SSA-2 (Smallest Space Analysis 2) by Guttman (1968) and Lingoes (1973)

Key Words and Phrases ; analysis of asymmetry, MDS, asymmetric multidimensional scaling, GIPS-CAL, parallelogram, person perception, halo effect
* Department of Psychology, Aichigakuin University, Nisshin-cho, Aichi-gun, Aichi, 470-01 Japan
Parts of this paper were presented at the 7th and 17th annual meeting of the Behaviormetric Society of Japan, and at the Spring Meeting of the Psychometric Society in 1980.

imposes row compatibility and column compatibility on data and yields two solutions in a metric space. Tobler's wind model explains asymmetries by the direction of wind assigned to mesh points on the configuration of objects (Tobler, 1976-77). Sato's model represents asymmetries using Randers' metric which is an asymmetric metric function (Sato, 1988, 1989).

Somewhat different or completely different approaches have been taken by some researchers. The future matching model proposed by Tversky (1977) represents the similarity or dissimilarity between two objects by a linear combination of the measures of the common and distinctive features of the two objects.

Chino (1978, 1980) has proposed a model involving a geometric generalization of scalar products. This model fits the scalar (inner) product and the magnitude of the cross product of solution vectors to the symmetric and skew-symmetric parts of the data, respectively. However, the model was defined only for two or three dimensions, first.

Canonical analysis of asymmetry proposed by Constantine and Gower (Gower, 1977, Constantine and Gower, 1978) decomposes dissimilarity data matrix into two parts, the symmetric and skew-symmetric parts, and then analyzes them separately. While the symmetric part is analyzed by some established distance model, the skew-symmetric part is analyzed by its canonical decomposition. Asymmetric relationships are interpreted in terms of areas of triangles and colinearities. Therefore, the canonical analysis has a stroung resemblance to the model proposed by Chino.

DEDICOM proposed by Harshman (1978, 1982) can be said to be a more general inner product model than the model proposed by Chino. DEDICOM decomposes observed similarity data into some combination of few basic underlying relationships.

Of concern here is the model formulated by Chino (1978). This model is written as:

$$S_{jk} = a\boldsymbol{x}_j'\boldsymbol{x}_k + b\boldsymbol{x}_j'\boldsymbol{I}^*\boldsymbol{x}_k + c + E_{jk}, \qquad (1)$$

where the quantity S_{jk} is the observed similarity between objects O_j and O_k (similarity from O_j to O_k to be precise), while the matrix $\boldsymbol{I}^*$ is a special skew-symmetric matrix of form

$$\boldsymbol{I}^* = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{I}^* = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad (2)$$

for two and three dimensions, respectively. Quantities, a, b, and c are unknown constants.

The quantity, $\boldsymbol{x}_j'\boldsymbol{I}^*\boldsymbol{x}_k$ in the second right-hand term of Eq. (1) has a special meaning. In the two-dimensional case, it becomes

$$\boldsymbol{x}_j'\boldsymbol{I}^*\boldsymbol{x}_k = x_{j1}x_{k2} - x_{j2}x_{k1}, \qquad (3)$$

which is just the cross product of two vectors $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$, the absolute value of which represents the *area of the parallelogram* spanned by the two vectors. It should be noted that the cross product in a two-dimensional space is a *scalar*.

The sign of the term $\boldsymbol{x}_j'\boldsymbol{I}^*\boldsymbol{x}_k$ determines the orientation of the parallelogram. This orientation can be easily guessed by defining the *positive direction* in the one-two plane in the following way : let us choose the horizontal and vertical axes, respectively , as the first and second axes, and suppose that the sign of b in Eq. (1) is positive. Then we say that the positive direction in this plane is *counterclockwise*. If the sign of b is negative, we then say that the positive direction is *clockwise*. The orientation of the parallelogram is positive if the sense (counterclockwise or clockwise) in which the vertices formed by the tails of the two vectors follow each other is the same as that for the positive direction in the plane. The orientation is negative if this sense is different from that for the positive direction. The positive direction therefore indicates the orientation of asymmetry in the plane.

In the three-dimensional case, the quantity becomes

$$\boldsymbol{x}_j'\boldsymbol{I}^*\boldsymbol{x}_k=(x_{j2}x_{k3}-x_{j3}x_{k2})+(x_{j3}x_{k1}-x_{j1}x_{k3})+(x_{j1}x_{k2}-x_{j2}x_{k1}). \quad (4)$$

In this case, the cross product or vector product of two vectors $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ is a *vector* and its first, second, and third components are, respectively, the first, second, and third right-hand terms of Eq. (4) in a right-handed coordinate system. It is evident, from the forms of these terms in Eq. (4), that the absolute values of these terms represent areas of parallelograms in two-dimensional spaces. In fact, the first, second, and third terms correspond, respectively, to parallelograms projected onto the x_2-x_3, x_3-x_1, x_1-x_2 planes of the parallelogram spanned in space by the vectors $\boldsymbol{x}_j, \boldsymbol{x}_k$. Fig. 1 .1 illustrates this. This means that the geometrical interpretation of the quantity in three dimensions can be reducible to that of the quantity in two dimensions.

As suggested in Eq. (4), the positive directions in these planes are all counterclockwise if we choose the horizontal and vertical axes, respectively, as the second and third, the third and first, and the first and second axes when the sign of b is positive. If we exchange the horizontal and vertical axes of, say, the third-first plane, the positive direction must also be changed. As we did in the two-dimensional case, if the sign of b in Eq. (1) is negative, all the positive directions in those planes should be altered accordingly.

One short-coming of this model is that it was defined only for two or three dimensions. Chino (1979, 1980) has extended this model into higher dimensions. However, no discussion has been made concerning theoretical and practical implications of the extension. The current paper is devoted to this purpose.

In the next section, we shall introduce the extended model and discuss the theoretical implications of this model. We shall call the extended model GIPSCAL (a Generalized Inner Product multidimensional SCALing). In higher dimensions we can define neither the cross product nor the vector product. However, it will be shown that we can still define areas of parallelograms in higher dimensions and associate with the two vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ an area of a parallelogram, the square of which is a well-known quantity in numerical analysis.

In the third section, we shall refer to a psychological justification of the model. In the fourth section, we shall discuss the problem of double centering transformation. In the classical MDS by Torgerson (1958), this transformation plays a fundamental role prior to the factoring of an inner product matrix. The double centering transformation of the observed similarity matrix is contrasted with that of the squared distance matrix in the classical MDS. In the fifth section, we shall show some examples of application. In the discussion section, we shall show the necessity for developing a general maximum likeli-

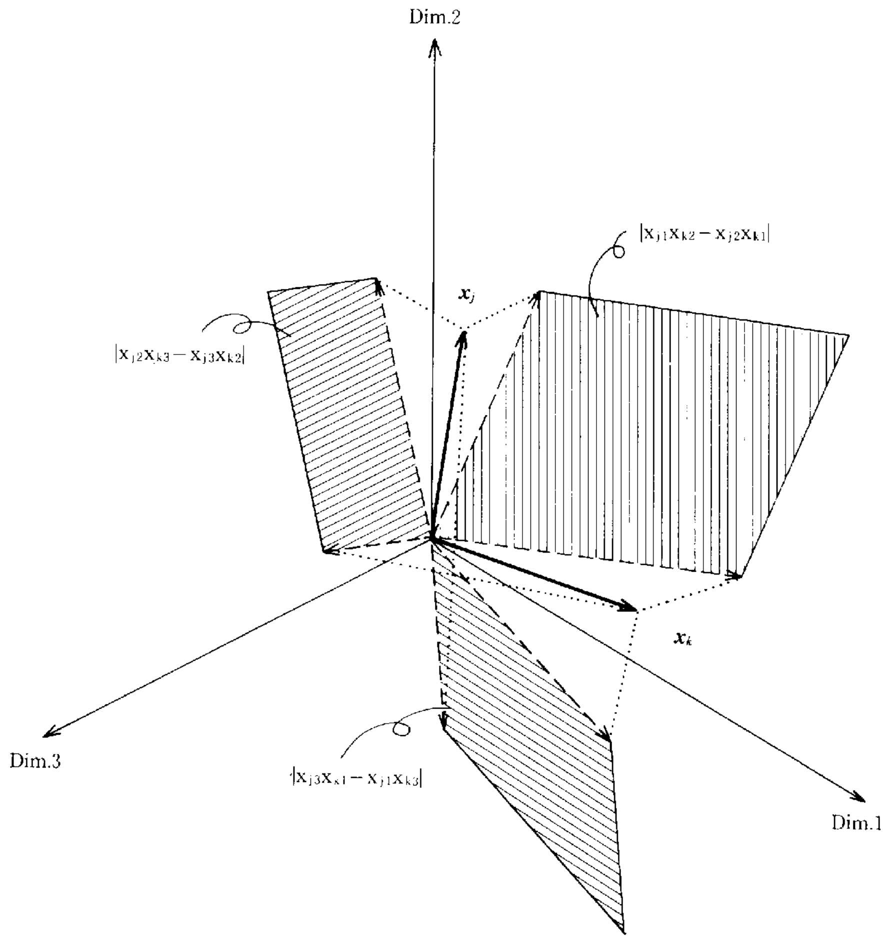

Fig. 1.1 Parallelograms projected onto three planes of the parallelogram spanned in space by the vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$.

hood MDS which enables us to select the best model out of a variety of extant asymmetric MDS's.

2. The model

Let the data be for N objects, say O_1, O_2, ..., O_N, which may be physical stimuli, classmates, companies, or nations and so forth. We shall denote the observed similarity from O_j to O_k by S_{jk}, assuming that the similarity is defined at the interval level of measurement in Stevens' terminology. In this sense, our model is a metric multidimensional scaling. The GIPSCAL model is written as:

$$S_{jk}=a\ \boldsymbol{x}_j'\ \boldsymbol{x}_k+b\ \boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k+c+E_{jk}, \tag{5}$$

where a, b, and c are unknown constants, and E_{jk} is an error term, while vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ are coordinates of objects in a p-dimensional Euclidean space, which are estimated from similarity data, S_{jk}'s. The matrix, $\boldsymbol{I}_p^*$ is a special skew-symmetric matrix of form,

$$\boldsymbol{I}_p^* = \left\{ sgn \begin{pmatrix} 1 & 2 & \cdots & p \\ \cdots & l & m & \cdots \end{pmatrix} \right\}, \tag{6}$$

where

$$sgn \begin{pmatrix} 1 & 2 & \cdots & p \\ \cdots & l & m & \cdots \end{pmatrix} = \begin{cases} 0, & \text{if the two indices } l,\ m \text{ are the same,} \\ 1, & \text{if } (..l\ m..) \text{ is an even permutation of } (1\ 2...p), \\ -1, & \text{if } (..l\ m..) \text{ is an odd permutation of } (1\ 2...p). \end{cases}$$

Of course, in the special case when p is less than four, $\boldsymbol{I}_2^*$ and $\boldsymbol{I}_3^*$ coincide with the corresponding matrices $\boldsymbol{I}^*$ of Chino's ASYMSCAL in two and three dimensions, respectively.

In this way, GIPSCAL is thought of as a natural extension of Chino's ASYMSCAL into higher dimensions, at least formally. However, at this point some questions may arise: is it possible in higher dimensions to consider the quantity $\boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k$ as the sum of the parallelograms projected onto two-dimensional planes? Can we associate with the two vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ a third vector uniquely? For, only in three dimensions can we define a vector product $\boldsymbol{x}_j \times \boldsymbol{x}_k$ that again is a vector.

The answer to the first question is yes. Quantity, $\boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k$ in the extended model has a very desirable property that, on the l-m plane,

$$\boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k = \sum_{l \neq m}^{p} sgn \begin{pmatrix} 1 & 2 & \cdots & p \\ \cdots & l & m & \cdots \end{pmatrix} x_{jl}\ x_{km}, \tag{7}$$

becomes $\quad sgn \begin{pmatrix} 1 & 2 & \cdots & p \\ \cdots & l & m & \cdots \end{pmatrix} (x_{jl}\ x_{km} - x_{jm}\ x_{kl}).$

This means that the geometrical interpretation of the quantity in higher dimensions is reducible to that of the area of the parallelogram spanned by vectors $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ in two dimensions. Fig. 1. 1 is a special case when $p=3$. In the case when the sign of b in Eq. (1) is positive, the positive direction in the l-m plane becomes counterclockwise if we choose the horizontal and vertical axes, respectively, as the l-th and m-th axes and if the permutation $(..l\ m..)$ is even.

The answer to the second question is no. As discussed in the previous section, in higher dimensions we cannot associate with the two vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ a third vector outside the plane spanned by the two vectors in a geometrical fashion. However, we can still associate with the two vectors the area of a parallelogram spanned by these vectors.

It is interesting to note that the square of the area is the *Gramian* or *Gram determinant* of the two vectors, which is well-known in numerical analysis and matrix algebra (Courant and John, 1974; Lancaster and Tismenetsky, 1985).

The Gramian $g\ (\boldsymbol{x}_j, \boldsymbol{x}_k)$ is written as

$$\begin{aligned} g\ (\boldsymbol{x}_j, \boldsymbol{x}_k) &= (\boldsymbol{x}_j'\ \boldsymbol{x}_j)\ (\boldsymbol{x}_k'\ \boldsymbol{x}_k) - (\boldsymbol{x}_j'\ \boldsymbol{x}_k)\ (\boldsymbol{x}_k'\ \boldsymbol{x}_j) \\ &= \begin{vmatrix} \boldsymbol{x}_j'\ \boldsymbol{x}_j, & \boldsymbol{x}_j'\ \boldsymbol{x}_k \\ \boldsymbol{x}_k'\ \boldsymbol{x}_j, & \boldsymbol{x}_k'\ \boldsymbol{x}_k \end{vmatrix}, \end{aligned} \tag{8}$$

It can also be shown readily that the following relation holds between the Gramian of the two vectors $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ and the right-hand term of Eq. (7):

$$g\ (\boldsymbol{x}_j,\ \boldsymbol{x}_k)=\sum_{\substack{l,m=1\\ l<m}}^{p}(x_{jl}\ x_{km}-x_{jm}\ x_{kl})^2, \tag{9}$$

Moreover, it is interesting to note that in general

$$g\ (\boldsymbol{x}_j,\ \boldsymbol{x}_k)\geqq 0, \tag{10}$$

and that equality holds only if $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ are *linearly dependent.*

In any case, we can say that GIPSCAL fits the scalar product (inner product) and the area of the parallelogram spanned by two vectors to the symmetric and asymmetric part of observed similarity judgements, respectively.

3. Psychological justification of the model

One may think that the availability of GIPSCAL is only in the parsimonious geometrical representation of the structure in the data. However, GIPSCAL has another availability in that it provides a psychological justification of why asymmetry arises.

For convenience, let us rewrite Eq. (5) as

$$S_{jk}=\boldsymbol{x}_j'\ \tilde{\boldsymbol{x}}_k+c+E_{jk}, \tag{11}$$

or

$$S_{jkr}=\boldsymbol{x}_j'\tilde{\boldsymbol{x}}_k+c+E_{jkr}, \tag{12}$$

where

$$\tilde{\boldsymbol{x}}_k=(a\boldsymbol{I}+b\boldsymbol{I}_p{}^*)\ \boldsymbol{x}_k=A\ \boldsymbol{x}_k, \tag{13}$$

Here S_{jkr} denotes the similarity from O_j to O_k rated by judge J_r.

Both Eq. (11) and Eq. (12) tell us that perception of O_k by O_j is distorted such that

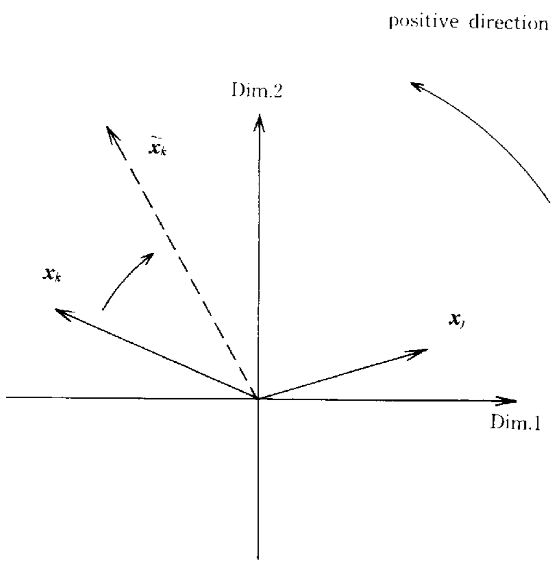

Fig. 3.1 Perceived image $\tilde{x}_k$ of member O_k by member O_j

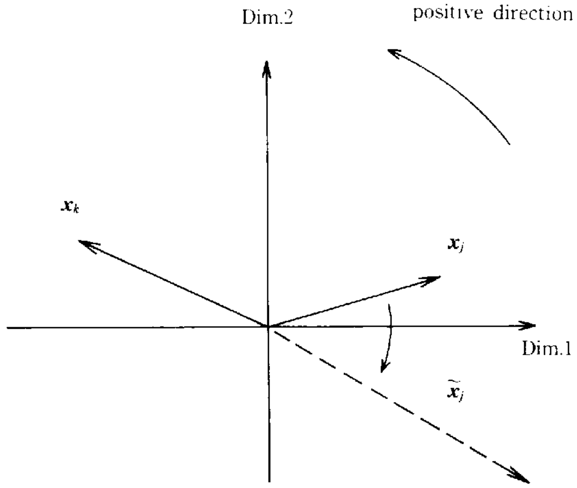

Fig. 3.2 Perceived image $\tilde{x}_j$ of member O_j by member O_k

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{A}\ \boldsymbol{x}_k, \tag{14}$$

where vector $\tilde{\boldsymbol{x}}_k$ is the perceived *image* of O_k, while $\boldsymbol{x}_k$ is the *preimage* of O_k. Figs. 3.1 and 3. 2 illustrate this. In this sense, the derived configuration by GIPSCAL is said to be a *preimage configuration*. In the field of person perception in social psychology, several general judgmental effects such as the *halo effect* and *logical error* have been known (Tagiuri, 1969). These effects may be combined to give rise to the perceived images discussed above.

It is of interest to note that the magnitude of the preimage of each object is related to the similarity to itself because, from Eq. (5)

$$S_{jj} = a\boldsymbol{x}_j{}'\boldsymbol{x}_j + c + E_{jj} \tag{15}$$

4. Double centering transformation as a preprocessing of original data

Coordinates of the derived configuration by MDS techniques are arbitrary in the sense of being only defined up to a central dilation, a translation (origin shift), and a rotation. By contrast, coordinates obtained by GIPSCAL are not arbitrary for the origin shift. However, we can have a configuration whose centroid is the origin in the special case when the double centering transformation is permitted to an original *full* similarity data matrix (we shall call such a configuration *origin centroid configuration*). By the full similarity data matrix we mean the similarity data matrix $\boldsymbol{S}$ which has no missing observations. To show this, let us first rewrite Eq. (5) in matrix form as

$$\boldsymbol{S} = a\boldsymbol{X}\boldsymbol{X}' + b\boldsymbol{X}\boldsymbol{I}_p{}^*\ \boldsymbol{X}' + c\ \boldsymbol{J}_N + \boldsymbol{E}. \tag{16}$$

where $\boldsymbol{S}$ is an N by N similarity data matrix, $\boldsymbol{X}$ is the preimage configuration, $\boldsymbol{J}_N$ is an N by N matrix whose entities are all 1, and $\boldsymbol{E}$ is an N by N matrix of error terms.

Second, we take note that the origin centroid configuration is obtained by premultiplying $\boldsymbol{X}$ by a well-known centering matrix (Horst, 1965)

$$\boldsymbol{G} = \boldsymbol{I} - (\boldsymbol{1}_N\ \boldsymbol{1}_N{}'/N). \tag{17}$$

The matrix $\boldsymbol{G}$ has the following properties:

$$\boldsymbol{G}'=\boldsymbol{G},\ \boldsymbol{G}^2=\boldsymbol{G},\ \boldsymbol{G}\ \mathbf{1}_N=\mathbf{0}, \tag{18}$$

and is an orthogonal projection matrix. Here, $\mathbf{1}_N$ is an N-dimensional vector whose components all equal unity.

Then, by a double centering transformation of the matrix $\boldsymbol{S}$, Eq. (16) can be rewritten as

$$\begin{aligned}\boldsymbol{S}_c &= \boldsymbol{G}\ \boldsymbol{S}\ \boldsymbol{G}' \\ &= a\boldsymbol{X}_c\ \boldsymbol{X}_c' + b\boldsymbol{X}_c\boldsymbol{I}_p^*\ \boldsymbol{X}_c' + c\boldsymbol{G}\boldsymbol{J}_N\ \boldsymbol{G}' + \boldsymbol{G}\ \boldsymbol{E}\ \boldsymbol{G}',\end{aligned} \tag{19}$$

where the matrix $\boldsymbol{X}_c$ is the origin centroid configuration. For the original full similarity data matrix, the third right-hand term of Eq. (19) is zero, since

$$\boldsymbol{G}\ \boldsymbol{J}_N\ \boldsymbol{G}'=\boldsymbol{O}. \tag{20}$$

Here $\boldsymbol{O}$ is a matrix whose entities are all 0.

It is interesting to note that the Young-Householder transformation is written as

$$\begin{aligned}\boldsymbol{P}_c &= \boldsymbol{X}_c\ \boldsymbol{X}_c' = \boldsymbol{G}\ \boldsymbol{P}\ \boldsymbol{G}' = -\boldsymbol{G}\ \boldsymbol{T}\ \boldsymbol{G}'/2, \\ &= \{-(d_{jk}^2 - \bar{d}_{j\cdot}^2 - \bar{d}_{\cdot k}^2 + \bar{d}_{\cdot\cdot}^2)/2\},\end{aligned} \tag{21}$$

and that

$$\boldsymbol{G}\ \boldsymbol{S}\ \boldsymbol{G}' = \{S_{jk} - \bar{S}_{j\cdot} - \bar{S}_{\cdot k} + \bar{S}_{\cdot\cdot}\}, \tag{22}$$

where matrices $\boldsymbol{P}$ and $\boldsymbol{T}$ are an inner product matrix and a squared distance matrix, respectively.

It is clear from Eqs. (19) and (21) that double centering transformation of the similarity data matrix $\boldsymbol{S}$ leads to the following equation:

$$\boldsymbol{S}_c = -a\boldsymbol{G}\ \boldsymbol{T}\ \boldsymbol{G}'/2 + b\boldsymbol{X}_c\ \boldsymbol{I}_p^*\ \boldsymbol{X}_c' + c\ \boldsymbol{G}\ \boldsymbol{J}_N\ \boldsymbol{G}' + \boldsymbol{G}\ \boldsymbol{E}\ \boldsymbol{G}'. \tag{23}$$

Eq. (23) describes the relation between similarity and distance in GIPSCAL. To be precise, it states that the doubly centered similarity is a linear combination of the squared distance between two vectors and the area of the parallelogram spanned by the two vectors except for constant and error terms in GIPSCAL.

To this point we have assumed the full similarity data matrix. However, in many cases we face data matrices with missing observations. In such cases, GIPSCAL provides near origin centroid configuration under the double centering transformation. This will be explained in the following section.

Some readers may doubt whether the double centering transformation of the original similarity matrix is appropriate. In fact, this transformation sometimes provides a better fit to the GIPSCAL model, but sometimes also a worse fit. Apparently, the results depend on the data.

Aside from the discussion of goodness of fit, there is no logical reason why we must choose the origin centroid configuration or near origin centroid one in the case of the inner product model. Thus, in the recent version of the GIPSCAL programme, this transformation is only one of the options in the preprocessing of original data.

5. The algorithm

Suppose that we are given NT elements, of the S_{jk}.

Let

$$e_{jk}=\begin{cases}1, \text{ if } S_{jk} \text{ is given}\\ 0, \text{ if } S_{jk} \text{ is not given}\end{cases} \tag{24}$$

Then
$$NT=\sum_{j=1}^{N}\sum_{k=1}^{N} e_{jk} \tag{25}$$

A reasonable procedure would seem to be to minimize the following function Q in a least square sense:

$$Q=\sum_{j=1}^{N}\sum_{k=1}^{N} e_{jk}\,(S_{jk}-a\boldsymbol{x}_j'\,\boldsymbol{x}_k-b\,\boldsymbol{x}_j'\,\boldsymbol{I}_p^*\,\boldsymbol{x}_k-c)^2. \tag{26}$$

For attacking this problem, we have chosen a familiar alternating least squares method, which is the same algorithm taken as that in Chino's ASYMSCAL.

To minimize Q, we first take the derivative of Q of Eq. (26) with respect to $\boldsymbol{x}_h$, where $h\neq j,\ k$. Then, using the matrix $\boldsymbol{A}$ defined already in Eq. (13), we have

$$\begin{aligned}\partial Q/\partial \boldsymbol{x}_h=-2\{&\sum_{j=1}^{N} e_{jh}\,(S_{jh}-\boldsymbol{x}_j'\,\boldsymbol{A}\,\boldsymbol{x}_h-c)\,\boldsymbol{A}'\,\boldsymbol{x}_j\\ &+\sum_{k=1}^{N} e_{hk}\,(S_{hk}-\boldsymbol{x}_h'\,\boldsymbol{A}\,\boldsymbol{x}_k-c)\,\boldsymbol{A}\,\boldsymbol{x}_k\}.\end{aligned} \tag{27}$$

Switching notation of the second right-hand term of Eq. (27) from k to j for later convenience, we have

$$\begin{aligned}\partial Q/\partial \boldsymbol{x}_h=-2\{&\sum_{j=1}^{N} e_{jh}\,(S_{jh}-\boldsymbol{x}_j'\,\boldsymbol{A}\,\boldsymbol{x}_h-c)\,A'\boldsymbol{x}_j\\ &+\sum_{j=1}^{N} e_{hj}\,(S_{hj}-\boldsymbol{x}_h'\boldsymbol{A}\,\boldsymbol{x}_j-c)\,\boldsymbol{A}\boldsymbol{x}_j\}.\end{aligned} \tag{28}$$

Rearranging the right member of Eq. (28), we get

$$\begin{aligned}\partial Q/\partial \boldsymbol{x}_h=-2\sum_{j=1}^{N}\Big[&\{(e_{jh}\,S_{jh}\,\boldsymbol{A}'+e_{hj}\,S_{hj}\,\boldsymbol{A})-c\,(e_{jh}\,\boldsymbol{A}'+e_{hj}\,\boldsymbol{A})\}\,\boldsymbol{x}_j\\ &-(e_{jh}\,\boldsymbol{A}'\,\boldsymbol{x}_j\,\boldsymbol{x}_j'\,\boldsymbol{A}+e_{hj}\,\boldsymbol{A}\boldsymbol{x}_j\,\boldsymbol{x}_j'\boldsymbol{A}')\,\boldsymbol{x}_h\Big].\end{aligned} \tag{29}$$

Then, setting $\partial Q/\partial \boldsymbol{x}_h$ equal to zero, and rearranging, we find

$$\begin{aligned}(\boldsymbol{A}'\,\boldsymbol{Y}_h\,\boldsymbol{A}+\boldsymbol{A}\,\boldsymbol{Z}_h\,\boldsymbol{A}')\,\boldsymbol{X}_h=\sum_{j=1}^{N}&\{(e_{jh}\,S_{jh}\,\boldsymbol{A}'+e_{hj}\,S_{hj}\,\boldsymbol{A})\\ &-c\,(e_{jh}\,\boldsymbol{A}'+e_{hj}\,\boldsymbol{A})\}\,\boldsymbol{x}_j,\end{aligned} \tag{30}$$

where

$$\boldsymbol{Y}_h=\boldsymbol{X}'\boldsymbol{B}_h\,\boldsymbol{X},\qquad \boldsymbol{Z}_h=\boldsymbol{X}'\boldsymbol{D}_h\,\boldsymbol{X}, \tag{31}$$

and

$$\boldsymbol{B}_h=diag\,(e_{ih}),\qquad \boldsymbol{D}_h=diag\,(e_{hi}). \tag{32}$$

Finally we have

$$X_h = C_h^{-1} \ (A' u_h + A \ v_h), \tag{33}$$

where

$$C_h = A' Y_h \ A + A \ Z_h \ A', \tag{34}$$

$$u_h = \sum_{j=1}^{N} e_{jh} \ (S_{jh} - c) \ x_j, \tag{35}$$

$$v_h = \sum_{j=1}^{N} e_{hj} \ (S_{hj} - c) \ x_j. \tag{36}$$

The form (33) suggests the iterative solution

$$x_{h,\ t+1} = C_{h,t}^{\ 1} \ (A_t' u_{h,t} + A_t \ v_{h,t}). \tag{37}$$

The value $x_{j,t}$ obtained at trial t is used to obtain the next $x_{h,t+1}$.

At each trial t, the configuration is normalized by scaling each vector x_j so that the maximum $(|x_1|, |x_2|, \ldots, |x_N|)$ equals one. This is the same strategy as in Chino's ASYMSCAL.

Secondly, taking the derivatives of Q of Eq. (26) with respect to c, we have

$$\partial Q/\partial c = -2 \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ (S_{jk} - a x_j' \ x_k - b x_j' \ I_p^* \ x_k - c). \tag{38}$$

Then, setting $\partial Q/\partial c$ equal to zero, and solving for c, we find

$$c = \bar{S} - a\bar{P} - b \ \bar{O}, \tag{39}$$

where

$$\bar{S} = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ S_{jk}/NT, \tag{40}$$

$$\bar{P} = \sum_{j=1}^{N} \sum_{j=1}^{N} e_{jk} \ x_j' \ x_k/NT, \tag{41}$$

and

$$\bar{O} = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ x_j' \ I_p^* \ x_k/NT. \tag{42}$$

Next, substituting for c in Eq. (26), we have

$$Q = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ \{(S_{jk} - \bar{S}) - a \ (x_j' x_k - \bar{P}) \\ - b \ (x_j' \ I_p^* \ x_k - \bar{O})\}^2. \tag{43}$$

Taking the derivatives of Q with respect to a and b respectively, we have

$$\partial Q/\partial a = 2 \ NT \ (a \ \sigma_p^2 + b \ \sigma_{po}^2 - \sigma_{sp}), \tag{44}$$

and

$$\partial Q/\partial b = 2 \ NT \ (a \ \sigma_{po} - b\sigma_o^2 - \sigma_{so}), \tag{45}$$

where

$$\sigma_{sp} = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ (S_{jk} - \bar{S}) \ (x_j' x_k - \bar{P})/NT, \tag{46}$$

$$\sigma_{so} = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \ (S_{jk} - \bar{S}) \ (x_j' \ I_p^* \ x_k - \bar{O})/NT, \tag{47}$$

$$\sigma_{po}=\sum_{j=1}^{N}\sum_{k=1}^{N} e_{jk}(\boldsymbol{x}_j'\ \boldsymbol{x}_k-\bar{P})\ (\boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k-\bar{O})/NT, \tag{48}$$

$$\sigma_p{}^2=\sum_{j=1}^{N}\sum_{k=1}^{N} e_{jk}\ (\boldsymbol{x}_j'\boldsymbol{x}_k-\bar{P})^2/NT, \tag{49}$$

$$\sigma_o{}^2=\sum_{j=1}^{N}\sum_{k=1}^{N} e_{jk}\ (\boldsymbol{x}_j'\ \boldsymbol{I}_p^*\ \boldsymbol{x}_k-\bar{O})^2/NT. \tag{50}$$

Then, setting $\partial Q/\partial b$ and $\partial Q/\partial b$ equal to zero respectively, we find

$$a=(\sigma_{sp}\ \sigma_o{}^2-\sigma_{so}\ \sigma_{po})/(\sigma_p{}^2\ \sigma_o{}^2-\sigma_{po}{}^2), \tag{51}$$

$$b=(\sigma_{so}\ \sigma_p{}^2-\sigma_{sp}\ \sigma_{po})/(\sigma_p{}^2\ \sigma_o{}^2-\sigma_{po}{}^2). \tag{52}$$

We shall now discuss the problem of preprocessing the original similarity data. GIPSCAL has two options concerning this problem. One is the simple normalization of the original data, in which case S_{jk}'s are normalized in such a way that their mean and the standard deviation equal zero and one, respectively. The other is the double centering transformation. As pointed out in the previous section, in this case, the quantity c becomes zero if and only if the original data have no missing observations. Otherwise c is not equal to zero. For the right-hand term of Eq. (42) will not equal zero if we have asymmetric missing observations (i.e., either e_{jk} or e_{kj} is zero).

Next, we shall discuss the measure of goodness of fit. The total variance of S_{jk}, that is, $\sigma_s{}^2$ would be partitioned into two parts, with the aid of Eqs. (43), (51), (52). Thus, we have

$$\sigma_s{}^2=\sigma_s{}^2\ (r_{sp}{}^2+r_{so}^2-2r_{sp}\ r_{so}\ r_{po})/(1-r_{po}{}^2)+\sigma_e{}^2, \tag{53}$$

or

$$1=(r_{sp}{}^2+r_{so}{}^2-2r_{sp}\ r_{so}\ r_{po})/(1-r_{po}{}^2)+(\sigma_e/\sigma_s)^2, \tag{54}$$

where

$$r_{sp}=\sigma_{sp}/(\sigma_s\ \sigma_p), \tag{55}$$

$$r_{so}=\sigma_{so}/(\sigma_s\ \sigma_o), \tag{56}$$

$$r_{po}=\sigma_{po}/(\sigma_p\ \sigma_o), \tag{57}$$

and

$$\sigma_e{}^2=Q_{min}/NT. \tag{58}$$

We can define the first right-hand term of Eq. (54) as the indicator of goodness of fit to the model. Thus, it is defined to be

$$F=(r_{sp}{}^2+r_{so}{}^2-2r_{sp}\ r_{so}\ r_{po})/(1-r_{po}{}^2). \tag{59}$$

For the full similarity data matrix, we have

$$F=r_{sp}{}^2+r_{so}{}^2, \tag{60}$$

which coincides with the indicator as in Chino's ASYMSCAL.

Finally, we shall consider the problem of convergence. It is desirable to obtain the second-order derivative Hessian matrix in order to help evaluate whether an extremum of the function Q has been found and to implement the more efficient Newton-Raphson method.

Remembering Eq. (34), we obtain

$$\partial/\partial \boldsymbol{x}_h \ (\partial Q/\partial \boldsymbol{x}_h) = 2\boldsymbol{C}_h. \tag{61}$$

From the definition of $\boldsymbol{B}_h$ and $\boldsymbol{D}_h$, it is clear that $\boldsymbol{B}_h^2 = \boldsymbol{B}_h$ and $\boldsymbol{D}_h^2 = \boldsymbol{D}_h$. Hence

$$\boldsymbol{Y}_h = \boldsymbol{X}'\boldsymbol{B}_h \ \boldsymbol{X} = (\boldsymbol{B}_h \ \boldsymbol{X})' \ (\boldsymbol{B}_h \ \boldsymbol{X}), \tag{62}$$

and

$$\boldsymbol{Z}_h = \boldsymbol{X}'\boldsymbol{D}_h \ \boldsymbol{X} = (\boldsymbol{D}_h \ \boldsymbol{X})' \ (\boldsymbol{D}_h \ \boldsymbol{X}). \tag{63}$$

Rewriting the Hessian matrix $\boldsymbol{C}_h$ by using Eqs. (62) and (63), we have

$$\begin{aligned} \boldsymbol{C}_h &= \boldsymbol{A}' \ \boldsymbol{Y}_h \ \boldsymbol{A} + \boldsymbol{A} \ \boldsymbol{Z}_h \ \boldsymbol{A}', \\ &= (\boldsymbol{B}_h \ \boldsymbol{X} \ \boldsymbol{A})' \ (\boldsymbol{B}_h \ \boldsymbol{X} \ \boldsymbol{A}) + \ (\boldsymbol{D}_h \ \boldsymbol{X} \ \boldsymbol{A}')' \ (\boldsymbol{D}_h \ \boldsymbol{X} \ \boldsymbol{A}'), \\ &= \boldsymbol{P}_h' \ \boldsymbol{P}_h + \boldsymbol{Q}_h' \boldsymbol{Q}_h, \end{aligned} \tag{64}$$

where

$$P_h = \boldsymbol{B}_h \ \boldsymbol{X} \ \boldsymbol{A} \text{ and } \boldsymbol{Q}_h = \boldsymbol{D}_h \ \boldsymbol{X} \ \boldsymbol{A}'. \tag{65}$$

It is not difficult to verify that the Hessian $\boldsymbol{C}_h$ given by Eq. (64) is *non-negative definite.*

Moreover, it is apparent from Eqs. (64) and (65) that the Hessian is *positive definite* if all the ranks of the matrices $\boldsymbol{B}_h$, $\boldsymbol{D}_h$, $\boldsymbol{X}$, and $\boldsymbol{A}$ are p.

Considering the case in which the Hessian is not positive definite, we recommend starting the iterative process of GIPSCAL from a variety of different initial configurations.

6. Examples of application

We shall now show some examples of application. First, we will apply GIPSCAL to a set of errorless data which was shown as an illustration in Chino (1978). We add

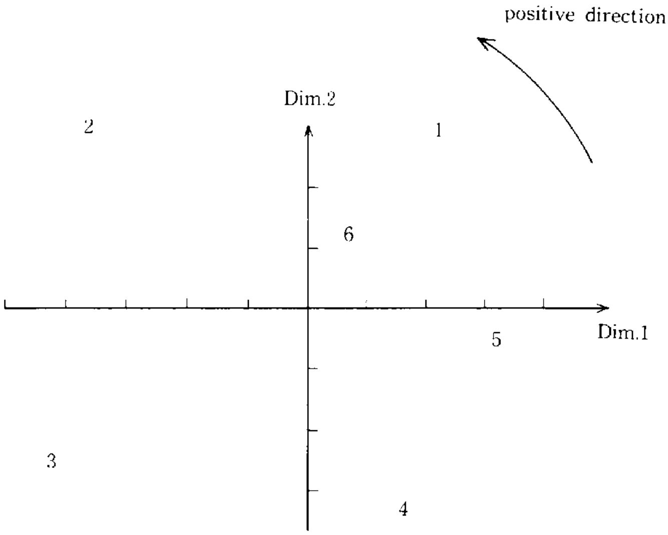

Fig. 6.1 Derived configuration for errorless data

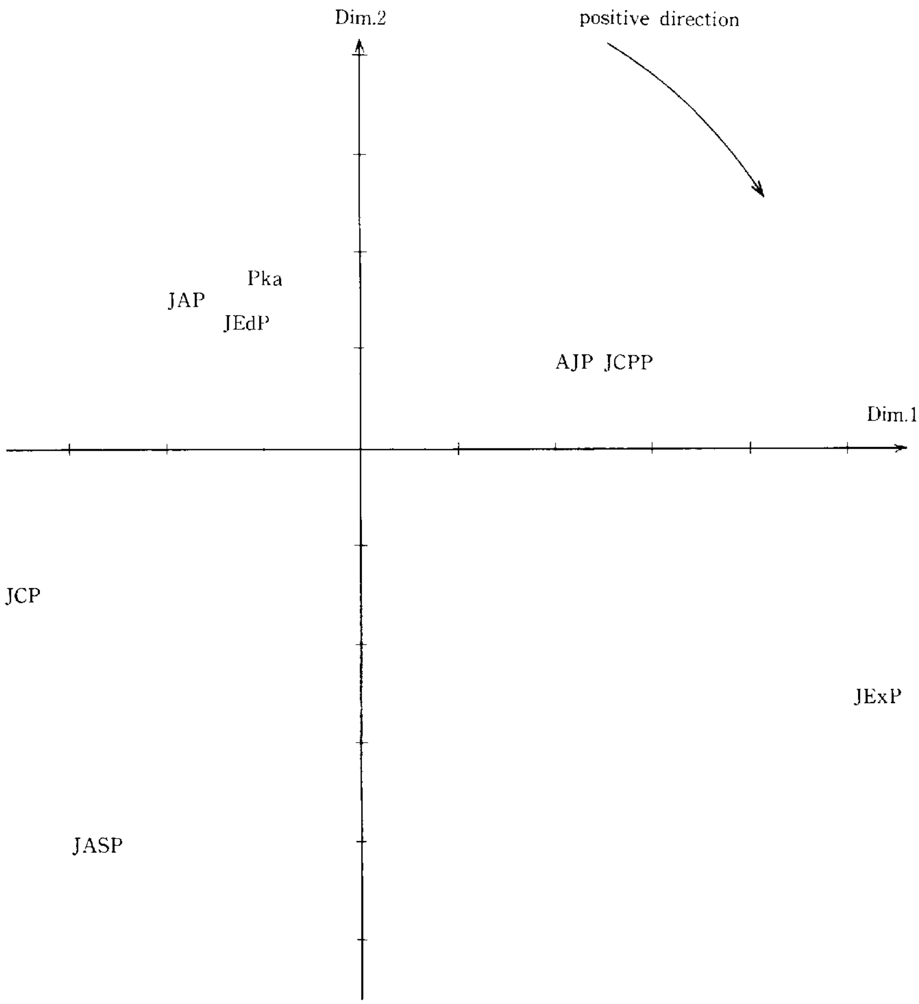

Fig. 6.2 The one-two plane for the journal citation data

diagonal elements to the data to make a full similarity data matrix. In this case both options of preprocessing, that is, the simple normalization and the double centering, recovered the assumed configuration of the data almost perfectly in two dimensions except for a rotation and/or a translation (origin shift). However, these options differed slightly in the coefficients r_{sp} and r_{so}. In the case of the simple normalization, r_{sp} and r_{so} were .720 and .693, respectively. In the case of the double centering, these were .715 and .698. As expected, the constant term c in GIPSCAL was almost zero in the case of the double centering, while it was not zero in the case of the simple normalization. Fig. 6. 1 shows the derived configuration for the double centering.

Second, we will show the results of application to journal citation data, especially the data shown in Table 2 in Chino (1978). Fig. 4 in that paper, of course, shows the derived configuration by Chino's ASYMSCAL for simply normalized data. The data were treated as similarity data. Values of F for doubly centered data were .835, .946, and .965 in two,

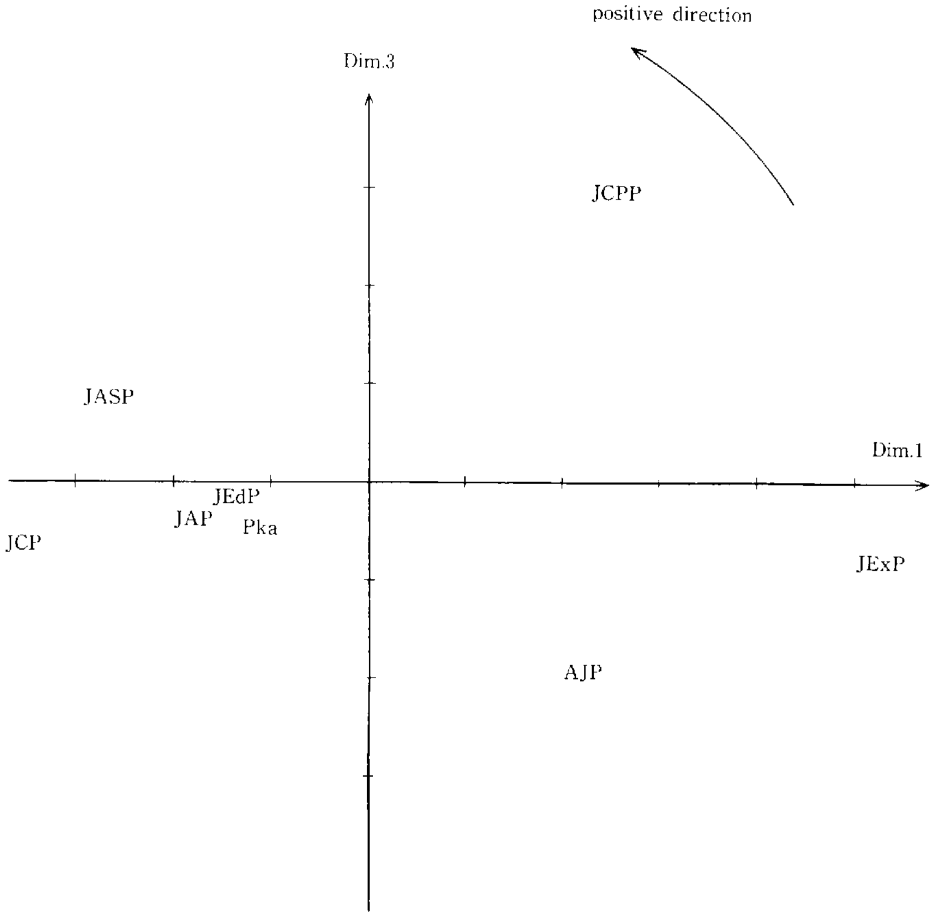

Fig. 6.3 The one-three plane for the journal citation data

three, and four dimensions, respectively.

We have chosen the three-dimensional configuration because the principal axis rotation of the four-dimensional configuration indicates that the latter configuration is almost degenerated. In other words, variation in the fourth dimension was very poor. Figs. 6.2 through 6.4 shows the planes defined by dimensions one and two, one and three, and two and three, respectively. We have not interpreted these principal axes because the number of journals is too small to assign them appropriate labels. In this case, r_{sp} and r_{so} were $-.907$ and $-.350$, respectively. As discussed in the introduction section, the sign of the constant b in our model, which is the same as that of r_{so}, determines the direction of asymmetry in each plane. Since the sign is negative, the positive directions in the planes defined by the above three are clockwise, counterclockwise, and clockwise, respectively. These directions are shown in the planes. It should be noted that the sign of r_{sp} is negative. This means that the shorter the distance between journals the less frequent the citation. For, we treated the data as similarity data.

The one-two plane explains the major symmetric relations contained in the original data. For example, locations of JAP, JEdP, and Pka indicate that citations between these

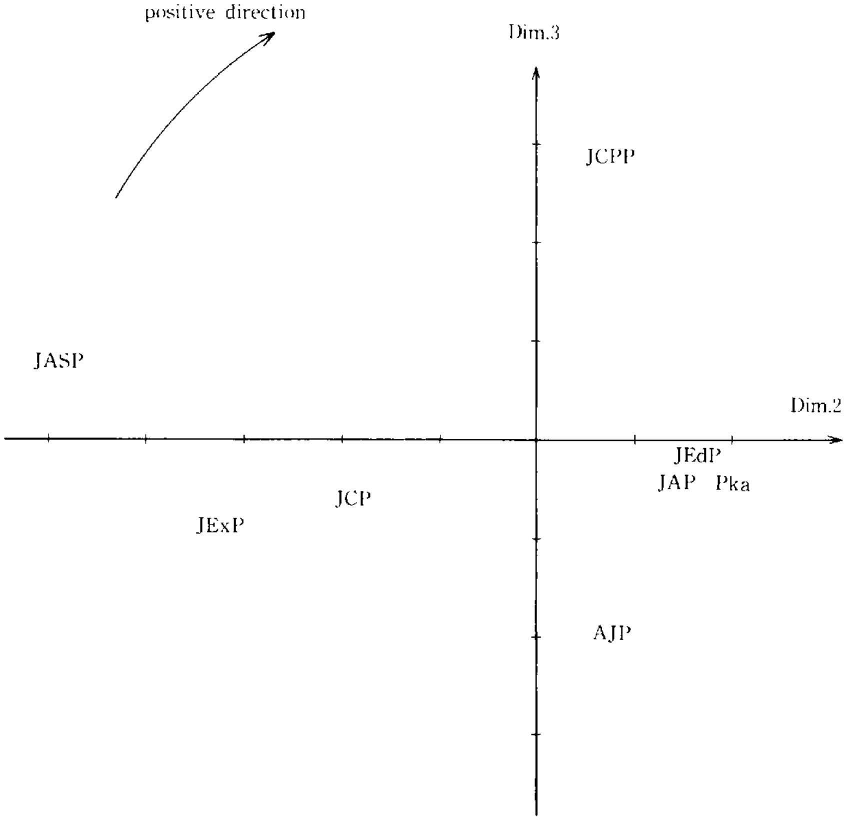

Fig. 6.4 The two-three plane for the journal citation data

journals are very poor. The same is true for AJP and JCPP. On the other hand, JExP is almost on the line passing through the origin and is located on the opposite side of journals JAP, JEdP, and Pka. This means that citations between JExP and each of these three journals are very frequent. Relation is the same between JCP and each of AJP and JCPP as well as between JASP and each of journals AJP and JCPP.

This plane also indicates the major asymmetric relations included in the data. For example, the positive direction in this plane describes the citation surpluses from JExP to JASP. The similar relations hold from JASP to JCP, from JCP to Pka, and so on. The only major asymmetry which contradicts the data is the relation between JCPP and JExP. This plane describes the citation surplus from JCPP to JExP, but it is not true.

The one-three plane draws the true citation surplus from JExP to JCPP. This can be readily confirmed if we notice that the positive direction in this plane is counterclockwise. This plane also emphasizes relatively poor citations between JASP, JAP, JCP, JEdP, and Pka and relatively frequent citations between JExP and each of the five journals.

The two-three plane might depict the major symmetric relations and minor asymmetric relations included in the data. For example, the positive direction in this plane (*i.e.*, clockwise) indicates the citation excesses from JCPP to each of JAP, JEdP, and Pka as well as from each of these three journals to AJP.

Finally, we will show the results of application to word association data gathered by

Table 6.1
Word association data gathered by Nakagawa (1986)

to / from	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1. label		5	2	3	3	2	1	4	4	5	2	3	6	4	2	3	1	6	2	3
2. fish	6		1	2	4	1	1	7	6	6	3	5	4	6	4	1	1	4	2	2
3. class	4	1		4	1	7	1	3	7	2	4	2	3	2	7	5	3	3	5	7
4. honor	4	2	4		1	5	1	2	4	2	4	1	1	2	6	2	4	3	3	3
5. relief	2	4	1	3		4	5	4	2	4	5	5	5	4	2	4	4	4	4	5
6. knight	2	3	4	6	4		2	3	2	3	3	1	1	4	7	3	2	4	5	2
7. jack	2	1	1	1	5	3		1	2	2	4	1	2	2	2	6	1	2	2	3
8. nest	4	6	2	1	2	2	1		2	4	3	4	2	7	5	2	1	4	3	3
9. yacht	5	7	5	3	2	2	2	3		2	4	1	4	5	5	4	3	6	6	2
10. plant	5	7	1	2	4	2	1	6	3		2	3	5	7	3	2	1	4	4	2
11. hope	3	4	4	3	6	4	4	2	5	4		4	5	2	6	2	2	2	3	5
12. mother	3	4	2	2	5	1	1	2	1	3	3		3	3	4	6	1	2	4	7
13. scale	6	3	4	2	4	1	2	2	5	4	2	3		5	4	6	3	7	3	4
14. tree	5	6	1	1	4	4	2	7	4	7	3	2	6		4	5	1	4	5	2
15. fight	2	4	7	6	3	7	4	6	5	3	6	4	4	6		4	4	3	5	7
16. iron	4	2	6	2	5	6	6	2	4	6	2	5	6	7	5		1	4	4	6
17. umpire	4	1	2	4	4	4	1	1	2	1	3	1	3	2	6	3		3	4	2
18. number	5	6	4	3	3	2	2	4	6	5	5	1	7	4	3	4	5		5	4
19. play	3	2	5	3	4	6	1	4	6	5	4	3	4	6	6	5	3	6		3
20. labor	4	2	7	3	4	2	4	3	5	1	4	7	7	2	7	6	1	4	3	

Nakagawa (1986). Table 6.1 shows this. The subject was asked to rate the magnitude of association by a seven point rating scale. Thus, the scores range from 1 to 7 for the weakest association and the strongest association, respectively. We analyzed the data as similarity data. Since the goodness of fit was poor in lower dimensions, we obtained solutions up to six dimensions. Values of F for doubly centered data were .461, .573, .672, .759, and .779, in two, three, four, five and six dimensions, respectively. Considering the deceleration of the improvement of the goodness of fit as well as ease of interpretation, we tentatively adopted the five-dimensional configuration. The principal axis solution has indicated that this configuration is not degenerated. Figs. 6.5 through 6.8 show the planes defined by dimensions one and two, one and three, one and four, and one and five, respectively. Although combinations of the five dimensions yield 10 planes, we selected the above four planes for simplicity as well as for space limitations. In this case, r_{sp} and r_{so} were .864 and $-.116$, respectively.

The first dimension might be labelled natural-social because the first four (*i.e.*, fish, plant, tree, nest) are related to natural resources and the last four (*i.e.*, class, labor, fight, and knight) to social resources or events. The second dimension has an even less obvious interpretation: the first three (that is, iron, labor, and mother) are associated with stable impressions and the last three (*i.e.*, fight, honor, and knight) with unstable impressions. In the same manner, we labelled the third, fourth, and fifth dimensions release-discrimination, soft-hard, and neutral-judge, respectively. Each plane defined by the combination of these

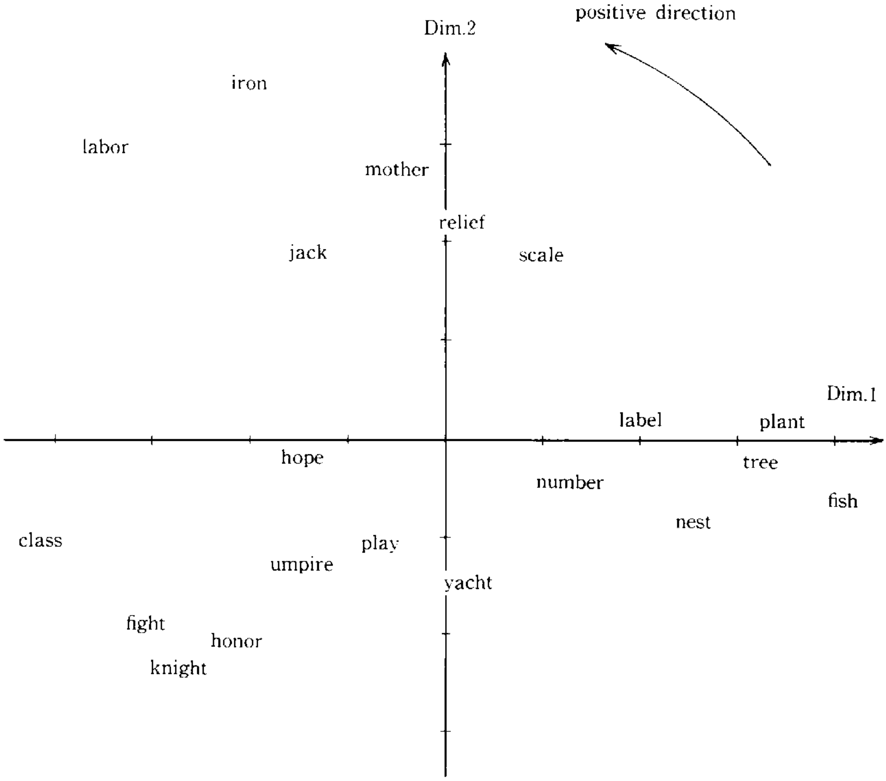

Fig. 6.5 The one-two plane for the word association data

dimensions describes various aspects of symmetric and asymmetric relations contained in the data. The positive directions in the above four planes are, respectively, counterclockwise, clockwise, counterclowkwise, and clockwise, since the sign of r_{so} is positive.

The one-two plane describes relatively symmetric associations between words within subgroups. For example, honor, knight, and fight form a subgroup. Plant and tree constitute another subgroup. In contrast with the distance model, the words located on the line passing through the origin indicate that they also have symmetric relations. Thus, jack and iron, for example, constitute a third subgroup. This plane also draws some aspects of asymmetric relation. The major ones are association imbalances between yacht and labor as well as knight and iron. The positive direction in this plane indicates, for example, that although the intensity of association with yacht from labor is strong, the reverse is relatively weak.

There are several major symmetric and asymmetric relations depicted in this plane which contradict the data. For example, intensity of association between class and labor is the same in the original data, but this plane draws a relatively large imbalance of association since the area of the parallelogram spanned by location vectors of these two words in relatively big. Another contradiction is the asymmetric relation between plant and iron. According to the original data, the intensity of association with plant from iron is stronger than that with iron from plant. However, the positive direction suggests the

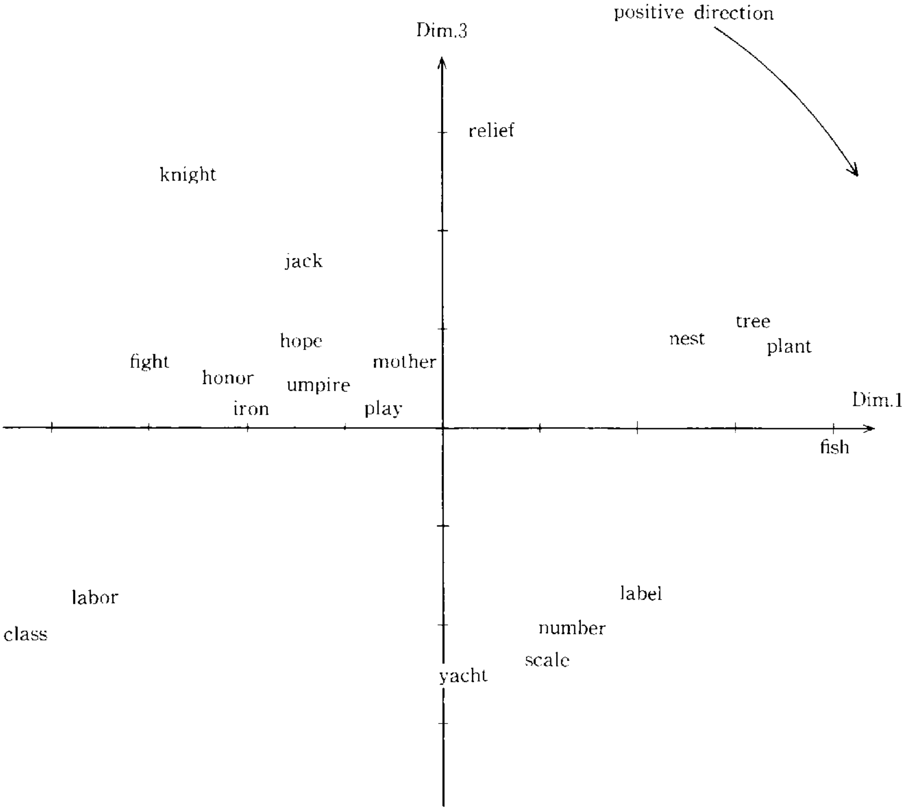

Fig. 6.6 The one-three plane for the word association data

reverse relation. In a similar manner, the relation indicated in this plane between words scale and labor contradicts the original data.

The one-three plane explains some of the relations which the one-two plane fails to describe. One is the symmetric relation between class and labor. Another is the asymmetric relation between relief and hope. This plane might mainly draw the asymmetric relation in the direction of the third dimension, that is, release-discrimination. The major three association imbalances in this direction are depicted between the pairs: class and knight, iron and knight, and hope and relief.

The one-four plane describes a major asymmetric relation between plant and iron, which neither the one-two plane nor the one-three plane can describe. That is, the positive direction in this plane describes the following relation: the intensity of association with plant from iron is strong, while the reverse weak.

The one-five plane describes major symmetric relations and minor asymmetric relations contained in the original data. As for symmetric relations, these include associations between words within subgroups: group 1 (fish and plant), group 2 (nest and tree), group 3 (class, fight, and labor), group 4 (scale and number). As for asymmetric associations, this plane describes many of the minor association imbalances between pairs nest and fight, tree and iron, label and tree, label and fish, umpire and number, honor and relief. The positive direction in this plane, which is clockwise, all coincides with the directions of association

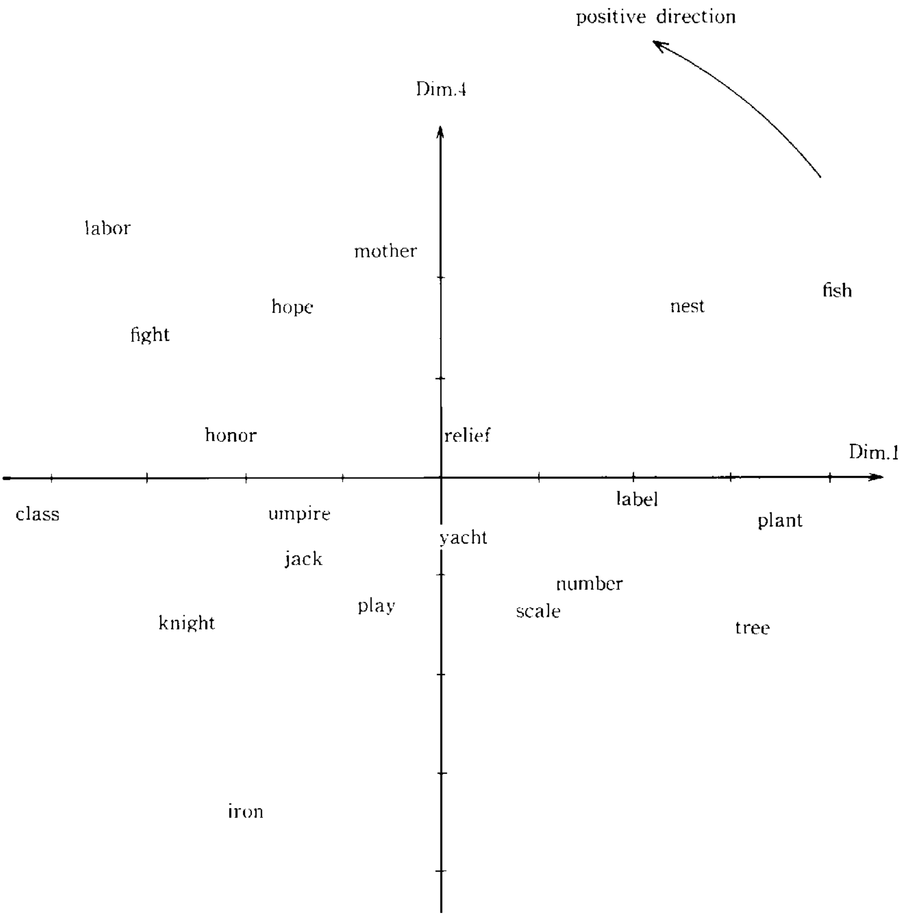

Fig. 6.7 The one-four plane for the word association data

imbalances between these pairs included in the original data.

7. Discussion

In this paper we have introduced an extension of Chino's ASYMSCAL into higher dimensions and discussed theoretical and practical implications of the extension. GIPSCAL, which is the extended model, fits the inner product and the area of the parallelogram spanned by two vectors to the symmetric and skew-symmetric parts of original or preprocessed similarity judgements, respectively. The square of the area of the parallelogram is nothing other than the Gramian of these two vectors.

GIPSCAL not only permits parsimonious geometrical representations of both the asymmetric structure and symmetric structure of data but also permits a social psychological justification as to why asymmetry arises.

Although inner product models are not in the mainstream of MDS, we have shown that our generalized inner product model-GIPSCAL can be related to a distance model indirectly, if we assume a double centering transformation as a preprocessing of original similarity data. Here it should be noted that the famous Young-Housholder transformation, which

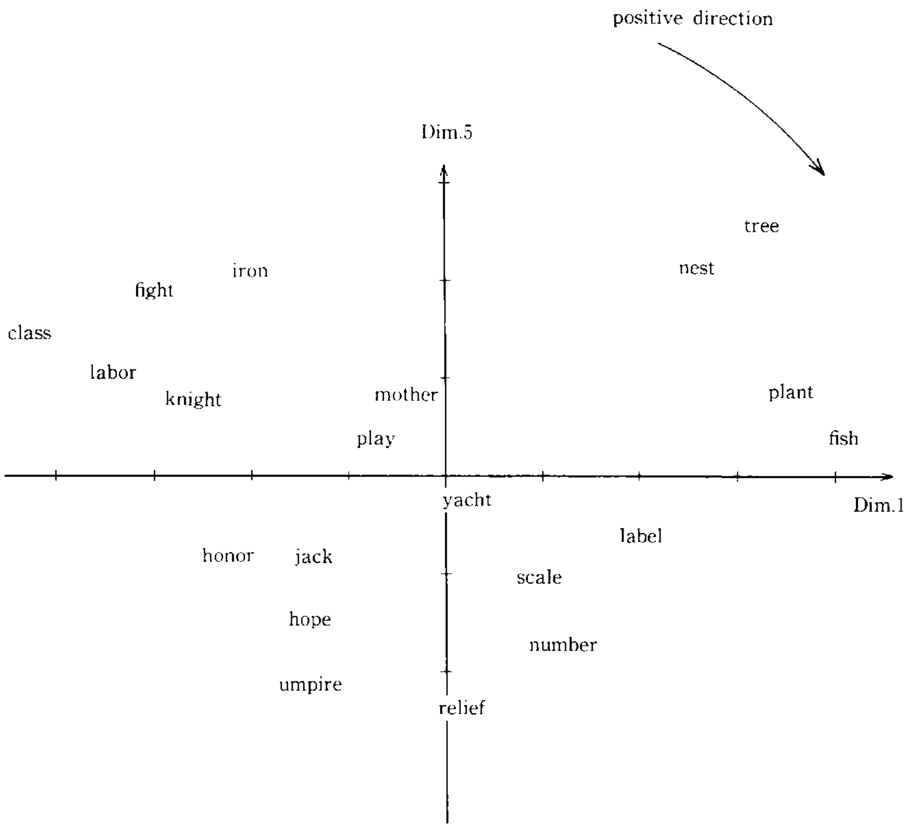

Fig. 6.8 The one-five plane for the word association data

plays a fundamental role in MDS techniques, is obtained through the double centering transformation of the inner product matrix produced from the configuration of objects whose origin is arbitrary.

One short-coming of GIPSCAL is its employment of a metric MDS. Considering the nature of data in social and behavioral sciences, it will be better to develop a nonmetric asymmetric MDS as Okada and Imaizumi (1987) have done.

Another short-coming is that GIPSCAL is merely a representation model, as are most of the extant MDS techniques. In the near future some researchers may develop *response models* which can take specific response processes in which asymmetry arises. However, since the context in which asymmetry arises seems to be diverse in character, we believe that there will be some important roles which representation models play. For example, if we face a set of longitudinal relational data matrices and consider the derived configuration obtained at each time by an appropriate extant MDS as a snapshot of a group formation process, it seems to be better to construct a group dynamics model which accounts for changes in group structure over time rather than to construct a specific response model which accounts for the cause of asymmetry at each time. In such a case, representation models may catch the structure at each time naturally without preoccupation with the specific response process assumed.

One other short-fall is that GIPSCAL as well as all the extant asymmetric MDS's does not utilize information about judgmental errors. To utilize such information, the maximum likelihood principle should be applied, as has been done in symmetric MDS (Ramsay, 1977, 1982 ; Takane, 1978a, 1978b, 1981 ; Takane and Carroll, 1981). This also enables us to compute AIC (Akaike, 1974), and therefore enables us to compare extant asymmetric MDS's and thereby to choose the best model.

Acknowledgements

The author wishes to express his thanks to anonymous referees for their constructive comments. Special thanks are due to Prof. W.B. Redekop for careful proofreading.

References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transaction on Automatic Control*, **19**, 716-723.

Baker, R.F., Young, F.W., and Takane, Y. (1977). An Asymmetric Euclidean model. *Paper submitted for publication to Psychometrika.*

Chino, N. (1978). A graphical technique for representing the asymmetric relationships between N objects. *Behaviormetrika*, **5**, 23-40.

Chino, N. (1979). Extension of Chino's ASYMSCAL into higher dimensions. *Paper presented at the 7th annual meeting of the Behaviormetric Society of Japan*, Osaka, Japan (in Japanese).

Chino, N. (1980). A unified geometrical interpretation of the MDS techniques for the analysis of asymmetry and related techniques. *Paper presented at the symposium on "Asymmetric multidimensional scaling" at the Spring Meeting of the Psychometric Society*, Iowa, U.S.A..

Constantine, A.G., and Gower, J.C. (1978). Graphical representation of asymmetric matrices. *Applied Statistics*, **27**, 297-304.

Courant, R. and John, F. (1974). *Introduction to Calculus and Analysis*, Vol. 2, New York : John Wiley & Sons.

Gower, J.C. (1977). The analysis of asymmetry and orthogonality. In J.R. Barra *et. al.*, (Eds.) *Recent Developments in Statistics.* New York : North-Holland Publishing Company.

Guttman, L. (1968). A general nonmetric technique for finding the smallest coordinate space for a configuration of points. *Psychometrika*, **33**, 469-506.

Harshman, R.A. (1978). Models for analysis of asymmetrical relationships among N objects or stimuli. *Paper presented at the First Joint Meeting of the Psychometric Society and The Society for Mathematical Psychology.* Hamilton, Canada.

Harshman, R.A., Green, P.E., Wind, Y., and Lundy M.E. (1982). A model for the analysis of asymmetric data in marketing research, *Marketing Science*, **1**, 205-242.

Hayashi, C. (1977). The problem of asymmetry in MDS. *Proceedings of the 45th annual conference of Japan Statistical Society*, Fukuoka, Japan (in Japanese).

Hayashi, C. (1977). The problem of asymmetry in MDS. *Proceedings of the 45th annual conference of Japan Statistical Society*, Fukuoka, Japan (in Japanese).

Horst, P. (1965). *Factor Analysis of Data Matrices*, New York : Holt, Rine-hart and Winston, Inc.

Krumhansl, C.L. (1978). Concerning the applicability of geometric models to similarity data : The interrelationship between similarity and spatial density. *Psychological Reviews*, **85**, 445-463.

Kruskal, J.B. (1964). Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis. *Psychometrika*, **29**, 1-28.

Lancaster, P. and Tismenetsky, M. (1985). *The Theory of Matrices*, 2nd Edition, New York : Academic Press.

Lingoes, J.C. (1973). *The Guttman-Lingoes Nonmetric Program Series.* Ann Arbor : Mathesis Press.

Nakagawa, M. (1986). Personal communication.

Okada, A. and Imaizumi, T. (1984). Geometric models for asymmetric similarity data. *Research Report*, Rikkyo (St. Paul's) University.

Okada, A. and Imaizumi, T. (1987). Nonmetric multidimensional scaling of asymmetric proximities. *Behaviormetrika*, **21**, 81-96.

Ramsay, J.O. Maximum likelihood estimation in multidimensional scaling. *Psychometrika*, 1977, **42**, 241-266.

Ramsay, J.O. (1982). Some statistical approaches to multidimensional scaling data. *The Journal of the Royal Statistical Society*, Series A (General), **145**, 285 312.

Saito, T. (1983). A method of multidimensional scaling to obtain a sphere configuration. *Hokkaido Behavioral Science Report*, Series-M, No. 4.

Saito, T. (1986). Multidimensional scaling to explore complex aspects in dissimilarity judgement. *Behaviormetrika*, **20**, 35-62.

Sato, Y. (1988). *Statistical Theory and Data Analysis II.* New York: North-Holland, 385-396.

Sato, Y. (1989). On distance representation using asymmetric metric function. *Paper presented at the 17th annual meeting of the Behaviormetric Society of Japan*, Okayama, Japan (in Japanese).

Tagiuri, R. (1969). Person perception. In G. Lindzey & E. Aronson (Eds.), *The Handbook of Social Psychology*, 2nd Edition, Vol. III, 395-449. Massachusetts: Addison-Wesley.

Takane, Y. (1978a). A maximum likelihood method for nonmetric multidimensional scaling: 1. The case in which all empirical pairwise orderings are independent-Theory. *Japanese Psychological Research*, **20**, 7-17.

Takane, Y. (1978b). A maximum likelihood method for nonmetric multidimensional scaling: 2. The case in which all empirical pairwise orderings are independent-Evaluation. *Japanese Psychological Research*, **20**, 105-114.

Takane,Y. (1981). Multidimensional successive categories scaling: A maximum likelihood method. *Psychometrika*, **46**, 9-28.

Takane, Y., and Carroll, D.J. (1981). Nonmetric maximum likelihood multidimensional scaling from directional rankings of similarities. *Psychometrika*, **46**, 389-405.

Tobler, W. (1976-77). Spatial interaction patterns. *Journal of Environmental Systems*, **6**, 271-301.

Torgerson, W.S. (1958). *Theory and Method of Scaling*, New York: Weley.

Tversky, A. (1977). Features of similarity. *Psychological Review*, **84**, 327-352.

Weeks, D.G. and Bentler, P.M. (1980). A restricted asymmetric multidimensional scaling model. *Paper presented at the Spring meeting of the Psychometric Society*, Iowa, U.S.A.

Weeks, D.G. and Bentler, P.M. (1982). Restricted multidimensional scaling models for asymmetric proximities. *Psychometrika*, **47**, 201-208.

Young, F.W. (1975). An asymmetric Euclidean model for multi-process asymmetric data. *Paper presented at U.S.-Japan Seminar on MDS*, San Diego.

Young, F.W. (1987). *Multidimensional Scaling: History, Theory, and Applications.* Hillsdale: Lawrense Erlbaum Associates.

(Received September, 1989. Revised November, 1989)